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A note on norm inequalities for positive operators

Jiewu Huang[1], Yang Peng[2] and Limin Zou[2*]

*Correspondence:
limin-zou@163.com
[2]School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, 404100, P.R. China
Full list of author information is available at the end of the article

Abstract

In this short note, we present a generalization of a norm inequality due to Bhatia and Kittaneh (Lett. Math. Phys. 43:225-231, 1998), which is also a refinement and a generalization of a result obtained by Kittaneh (Commun. Math. Phys. 104:307-310, 1986).

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1 Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. We shall always denote the singular values of A by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. Let $A, B \in M_n$ be Hermitian; the order relation $A \geq B$ means, as usual, that $A - B$ is positive semidefinite.

Let $A, B \in M_n$ be positive semidefinite. Bhatia and Kittaneh [1, Theorem 2.2] proved that for any positive integer m,

$$\left\|A^m + B^m\right\| \leq \left\|(A+B)^m\right\|. \tag{1.1}$$

Let $A, B \in M_n$. Kittaneh [2, Theorem 2.2] proved that

$$\|A+B\|_F^2 \leq \frac{1}{2}\left(\left\||A| + |B|\right\|_F^2 + \left\|\left|A^*\right| + \left|B^*\right|\right\|_F^2\right), \tag{1.2}$$

where $\|X\|_F$ is the Frobenius norm of X. For more information on the Schatten p-norm and its applications, the reader is referred to [3].

In this note, we present a generalization of inequality (1.1), which is also a refinement and a generalization of (1.2).

2 Main results

Now, we show the generalization of inequality (1.1).

Theorem 2.1 *Let $A, B \in M_n$ and suppose that p, q be real numbers with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for any positive integer m,*

$$\left\|A|A|^{m-1} + B|B|^{m-1}\right\| \leq \left\|\left(|A|^m + |B|^m\right)^{p/2}\right\|^{1/p} \left\|\left(\left|A^*\right|^m + \left|B^*\right|^m\right)^{q/2}\right\|^{1/q}. \tag{2.1}$$

Proof Let $A, B \in M_n$ with polar decompositions $A = U|A|$ and $B = V|B|$. It is known [4, p.15] that

$$\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} |A| & |A| \\ |A| & |A| \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U^* \end{bmatrix} \geq 0.$$

It follows that

$$\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix}^m + \begin{bmatrix} |B| & B^* \\ B & |B^*| \end{bmatrix}^m = 2^{m-1} \begin{bmatrix} |A|^m + |B|^m & |A|^m U^* + |B|^m V^* \\ U|A|^m + V|B|^m & |A^*|^m + |B^*|^m \end{bmatrix} \geq 0.$$

So, by Proposition 1.3.2 of [4, p.13], we have

$$U|A|^m + V|B|^m = \left(|A|^m + |B|^m\right)^{1/2} K \left(\left|A^*\right|^m + \left|B^*\right|^m\right)^{1/2}$$

for some contraction K. For $k = 1, \ldots, n$, by using Horn's inequality [5, p.72], we know that

$$\prod_{j=1}^{k} s_j\left(U|A|^m + V|B|^m\right) \leq \prod_{j=1}^{k} s_j^{1/2}\left(|A|^m + |B|^m\right) s_j^{1/2}\left(\left|A^*\right|^m + \left|B^*\right|^m\right). \tag{2.2}$$

Let

$$X = \operatorname{diag}\left(s_1^{1/2}\left(|A|^m + |B|^m\right), \ldots, s_n^{1/2}\left(|A|^m + |B|^m\right)\right)$$

and

$$Y = \operatorname{diag}\left(s_1^{1/2}\left(\left|A^*\right|^m + \left|B^*\right|^m\right), \ldots, s_n^{1/2}\left(\left|A^*\right|^m + \left|B^*\right|^m\right)\right).$$

Then inequality (2.2) is equivalent to

$$\prod_{j=1}^{k} s_j\left(U|A|^m + V|B|^m\right) \leq \prod_{j=1}^{k} s_j(XY).$$

Since weak log-majorization implies weak majorization, we get

$$\sum_{j=1}^{k} s_j\left(U|A|^m + V|B|^m\right) \leq \sum_{j=1}^{k} s_j(XY). \tag{2.3}$$

Thanks to the Fan dominance principle [5, p.93], we know that inequality (2.3) is equivalent to

$$\left\| U|A|^m + V|B|^m \right\| \leq \|XY\|. \tag{2.4}$$

By Hölder's inequality for unitarily invariant norms [5, p.95] (see also [6, Theorem 2.4]), we obtain

$$\|XY\| \leq \left\|X^p\right\|^{1/p} \left\|Y^q\right\|^{1/q}. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\|A|A|^{m-1}+B|B|^{m-1}\| \le \|(|A|^m+|B|^m)^{p/2}\|^{1/p}\|(|A^*|^m+|B^*|^m)^{q/2}\|^{1/q}.$$

This completes the proof. □

Remark 2.2 Inequality (2.1) is a norm version of the following scalar triangle inequality:

$$|z_1|z_1|^{m-1}+z_2|z_2|^{m-1}| \le |(|z_1|^m+|z_2|^m)^{p/2}|^{1/p}|(|z_1^*|^m+|z_2^*|^m)^{q/2}|^{1/q}.$$

Corollary 2.3 *Let $A, B \in M_n$. Then for any positive integer m,*

$$\|A|A|^{m-1}+B|B|^{m-1}\| \le \|(|A|+|B|)^m\|^{1/2}\|(|A^*|+|B^*|)^m\|^{1/2}. \tag{2.6}$$

Proof Putting $p = q = 2$ in inequality (2.1), we have

$$\|A|A|^{m-1}+B|B|^{m-1}\| \le \||A|^m+|B|^m\|^{1/2}\||A^*|^m+|B^*|^m\|^{1/2}.$$

It follows from inequality (1.1) and this last inequality that

$$\|A|A|^{m-1}+B|B|^{m-1}\| \le \|(|A|+|B|)^m\|^{1/2}\|(|A^*|+|B^*|)^m\|^{1/2}.$$

This completes the proof. □

Remark 2.4 If A and B are positive semidefinite, then by inequality (2.6), we get (1.1).

Remark 2.5 For $m = 1$, by inequality (2.1), we get

$$\|A+B\| \le \|(|A|+|B|)^{p/2}\|^{1/p}\|(|A^*|+|B^*|)^{q/2}\|^{1/q}.$$

In particular,

$$\|A+B\| \le \||A|+|B|\|^{1/2}\||A^*|+|B^*|\|^{1/2}, \tag{2.7}$$

which is a generalization and a refinement of inequality (1.2). For the usual operator norm, it is known that

$$\|A+B\|_\infty \le \sqrt{2}\||A|+|B|\|_\infty, \tag{2.8}$$

which is sharp. Since

$$\||A^*|+|B^*|\| \le \||A^*|\|+\||B^*|\| \le 2\||A|+|B|\|,$$

we find that inequality (2.7) is a strengthening of inequality (2.8).

Remark 2.6 Recently, Zou [7, Theorem 2.2] and Zou and He [8, Theorem 2.2] gave some generalizations of inequality (1.1) for normal matrices. Our result is different from the ones obtained by these authors.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by JH, YP, and LZ. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]College of Science, Guizhou Minzu University, Guiyang, Guizhou 550025, P.R. China. [2]School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, 404100, P.R. China.

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